Spring 2017 MATH5012

Real Analysis II

Solution to Exercise 2

Here μ is a Radon measure on $\mathbb{R}^n$. Many problems are taken from [R1].

(1) Use maximal function to give another proof of Lebesgue differentiation theorem. Setting

$$(T_r f)(x) = \frac{1}{\mu(\overline{B}_r(x))} \int_{\overline{B}_r(x)} |f - f(x)| d\mu \ ,$$

and

$$(Tf)(x) = \limsup_{r \to 0} (T_r f)(x) \ .$$

Show that $T_r f = 0$ μ-a.e.. Suggestion: For $\varepsilon > 0$, pick continuous g such that $\|f - g\|_{L^1} < \varepsilon$ and establish $Tf(x) \le Mh(x) + |h|(x)$ where $h = f - g$. Then use 7(a) in Ex 1.

Solution. Explained in class, or look up [R1].

(2) Let E be μ-measurable. Show that μ-a.e. $x \in \mathbb{R}^n \setminus E$ has density 0 in E.

Solution. Apply to the complement of the set and use the result on density 1.

(3) Let F be closed in $\mathbb{R}^n$ and $d(x, F)$ the distance from x to F,

$$d(x, F) = \inf \{|x - y| : y \in F\} \, .$$

(a) Show that

$$|d(x, F) - d(y, F)| \le |x - y| \ , \quad \forall x, y \in \mathbb{R}^n \ .$$

(b) Let x be a point of density 1 of $F \subset \mathbb{R}$. Show that

$$\frac{|d(y, F) - d(x, F)|}{|y - x|} \to 0 \text{ as } y \to x.$$

Solution. Note that I have modified this problem. First of all, as F is closed, for each $x \in \mathbb{R}^n$, there exists some $z \in F$ such that $d(x, F) = |x - z|$. Then

$$d(y, F) \leq |y - z| \leq |y - x| + |x - z| = |y - x| + d(x, F) \ ,$$

and so $d(y, F) - d(x, F) \leq |x - y|$. The full inequality follows from switching x and y.

Note. It is impressive that the distance function is always Lipschitz continuous with Lipschitz constant 1.

Next, take $n = 1$. Let x be a point of density 1 for F, so $d(x, F) = 0$ and it has zero density with respect to the complement of F, F'. For small $\varepsilon > 0$, there exists some δ_0 such that

$$\frac{\mathcal{L}^1(F' \cap [x - \delta, x + \delta])}{2\delta} < \varepsilon \ , \quad \forall 0 < \delta \leq \delta_0 \ .$$

We claim that for each $y = x + \delta, \delta \leq \delta_0$, $F \cap [y - \delta\varepsilon, y + \varepsilon\delta] \neq \phi$. For, if it is empty, that means $[y - \varepsilon\delta, y + \varepsilon\delta]$ is contained in F' so

$$\frac{\mathcal{L}^1(F' \cap [x - \delta, x + \delta])}{2\delta} \geq \frac{\mathcal{L}^1[y - \varepsilon\delta, y + \varepsilon\delta]}{2\delta} = \varepsilon \ ,$$

contradiction holds. It follows that

$$d(y, F) = d(x + \delta) \leq \varepsilon\delta \ ,$$

that is,

$$\frac{|d(x + \delta, F) - d(x, F)|}{\delta} \leq \varepsilon \ ,$$

and the conclusion follows. The same argument applies to the point $x - \delta$.

(4) For $\delta > 0$, let $I(\delta) = (-\delta, \delta)$. Given α and β, $0 \leq \alpha < \beta \leq 1$, construct a measurable set E so that the upper and lower limits of $\mathcal{L}^1(E \cap I(\delta))/2\delta$ are equal

to α and β respectively as $\delta \to 0$.

Solution. By reflecting about the origin if necessary, it suffices to consider the following function $f(\delta) = \mathcal{L}^1(E \cap [0, \delta))/\delta$, where $E \subseteq [0, \infty)$. For $0 < \alpha < \beta < 1$, let $r = \frac{\alpha}{\beta}(\frac{1-\beta}{1-\alpha}) \in (0,1)$ and $< \frac{\alpha}{\beta}$, $\gamma_n = r^n$ and $l_n = \gamma_n - \frac{\beta}{\alpha}\gamma_{n+1}$. Observe that l_n satisfies the following inequalities

$$\gamma_n - \gamma_{n+1} > l_n = \gamma_n - \frac{\beta}{\alpha}\gamma_{n+1} > 0.$$

Let E be $\bigcup_{n=1}^{\infty}[\gamma_n - l_n, \gamma_n]$. We first show that $f(\gamma_n) = \beta, \quad \forall n$,

$$\begin{aligned} \mathcal{L}^1(E \cap [0, \gamma_n)) = \sum_{k=n}^{\infty} l_k &= \sum_{k=n}^{\infty} \gamma_k - \frac{\beta}{\alpha} \sum_{k=n+1}^{\infty} \gamma_k \\ &= \gamma_n + \frac{\alpha - \beta}{\alpha} \sum_{k=n+1}^{\infty} \gamma_k = \beta\gamma_n \end{aligned}$$

Hence $f(\gamma_n) = \beta$. Next we will show that $f(\gamma_n - l_n) = \alpha$, by definition of l_n

$$\mathcal{L}^1(E \cap [0, \gamma_n - l_n)) = \beta\gamma_{n+1} = \alpha\gamma_n - \alpha l_n$$

we have $f(\gamma_n - l_n) = \alpha$. We try to show that f attains maximum and minimum at γ_n and $\gamma_n - l_n$ respectively. $\forall \delta \in [\gamma_{n+1}, \gamma_n - l_n]$, $\mathcal{L}^1(E \cap [0, \delta))$ is fixed, so f is decreasing on $[\gamma_{n+1}, \gamma_n - l_n]$. If $\delta \in [\gamma_n - l_n, \gamma_n]$,

$$f(\delta) = \frac{\beta\gamma_n - \gamma_n + \delta}{\delta} = 1 - (1-\beta)\frac{\gamma_n}{\delta}$$

we have f is increasing on $[\gamma_n - l_n, \gamma_n]$ and we have the following inequalities

$$\alpha \leq f(\delta) \leq \beta$$

with first equality holds when $\delta = \gamma_n - l_n$ and second equality holds when $\delta = \gamma_n$.

Result follows.

For the other cases (either $\alpha = 0$ or $\beta = 1$), we may consider 2 strictly monotonic sequences, $\alpha_k \downarrow \alpha$ and $\beta_k \uparrow \beta$ such that $\alpha_m < \beta_n, \forall n, m$. And let

$$r_k := \frac{\alpha_k}{2\beta_{k+1}} \left(\frac{1-\beta_k}{1-\alpha_{k+1}} \right) \text{ and } \gamma_{k+1} := r_k \gamma_k \text{ with } \gamma_1 = 1.$$

We immediately have $\gamma_k \to 0$ as $k \to \infty$ and

$$\gamma_{k+1} < \min\{\frac{\alpha_k}{\beta_{k+1}}, \frac{1-\beta_k}{1-\alpha_{k+1}}\}\gamma_k$$

With the above inequality, we may define $l_{2n-1} := \alpha_{2n-1}\gamma_{2n-1} - \beta_{2n}\gamma_{2n}$ and $l_{2n} := \beta_{2n}\gamma_{2n} - \alpha_{2n+1}\gamma_{2n+1}$ which satisfy

$$\gamma_n - \gamma_{n+1} > l_n > 0, \forall n,$$

$$\sum_{k=2n-1} l_k = \alpha_{2n-1}\gamma_{2n-1} \text{ and } \sum_{k=2n} l_k = \beta_{2n}\gamma_{2n}.$$

We may consider

$$E = \begin{cases} \bigcup_{n=1}^{\infty} [\gamma_n - l_n, \gamma_n] & \text{if } \alpha = 0 \\ \bigcup_{n=1}^{\infty} [\gamma_{n+1}, \gamma_{n+1} + l_n] & \text{if } \beta = 1 \end{cases}$$

then we have $f(\gamma_{2n-1}) = \alpha_{2n-1}$ and $f(\gamma_{2n}) = \beta_{2n}$. Result follows from similar arguments as before.

(5) If $A \subset \mathbb{R}^1$ and $B \subset \mathbb{R}^1$, define $A+B = \{a+b : a \in A, b \in B\}$. Suppose $m(a) > 0$, $m(b) > 0$. Prove that $A+B$ contains a segment, by completing the outline given in [R1].

Solution. Follow the hint in [R1].

(6) A point $x \in \mathbb{R}^n$ is called an atom for a measure λ if $\lambda(\{x\}) > 0$. Establish the

decomposition

$$\mu = f\mathcal{L}^n + \mu_{cs} + \sum_k a_k \delta_{x_k}, \quad a_k > 0,$$

where $f \in L^1(\mathcal{L}^n)$ and μ_{cs} has no atoms.

Solution. By Radon-Nikodym we have the decomposition $\mu = f\mathcal{L}^n + \mu_s$ where $\mu_s \perp \mathcal{L}^n$. Let $A_k = \{x : \ \mu_s(\{x\}) > 0, \ |x| \le k\}$ and $A = \bigcup_k A_k$. We claim that each A_k is a finite set. For let us pick N many points from A_k. We have

$$\infty > \mu_s(B_k(0)) \ge N \times \frac{1}{k} \ ,$$

which shows that N has an finite upper bound. Here we have used the fact that μ_s is Radon so that it is finite on balls. Now we know that A is a countable set $\{x_j\}$. Setting

$$\mu_d = \sum_j a_j \delta_{x_j} \ , \quad a_j = \mu_s(\{x_j\}) \ ,$$

the conclusion follows by letting $\mu_{cs} = \mu_s - \mu_d$.

(7) Let $\{x_n\}$ be an infinite sequence of distinct numbers in $[0, 1]$. Can you find an increasing function in $[0, 1]$ whose discontinuity set is precisely $\{x_n\}$?

Solution. Put $\mu = \sum 2^{-n}\delta_{x_n}$ where $R = \{x_n\}_{n=1}^\infty$. Define

$$F(x) = \mu(-\infty, x) = \sum_{x_k < x} \frac{1}{2^k} \ ,$$

be a function on $\mathbb{R}$. Now fix an $x \notin R$. Let $\varepsilon > 0$ be given. There exists N such that

$$\sum_N^\infty 2^{-n} < \varepsilon.$$

Then since $x \notin R$, we can choose $\delta > 0$ such that $x_1, \cdots, x_{N-1} \notin [x - \delta, x + \delta)$. Now whenever $x < y < x + \delta$,

$$F(y) - F(x) = \mu[x, y) \le \mu[x - \delta, x + \delta) < \varepsilon.$$

Similarly, we also have

$$F(x) - F(y) < \varepsilon$$

whenever $x - \delta < y < x$. Hence F is continuous outside R.

But for every $x \in R$, whenever $y > x$,

$$F(y) - F(x) = \mu[x, y) \geq 2^{-k}$$

for some k. This shows that F is not continuous at every point in R.

(8) (a) Consider the real line. Show that x is not an atom for μ if and only if its distribution function is continuous at x. Use (a) to construct a singular measure, that is, perpendicular to $\mathcal{L}^1$, without atoms. Suggestion: Consider the Cantor-Lebesgue function.

Solution. Refer to [R1]. This is an important example.

(9) Let μ be a singular measure with respect to $\mathcal{L}^1$ and f its distribution function. Show that for μ-a.e. x, either f'_+ or f'_- becomes ∞.

Solution. Let A^* be the support of μ. We know that $\mathcal{L}^1(A^*) = 0$ and $\mu(E) = \mu(E \cap A^*)$. Let

$$C_k = \{x : \underline{D}\mu(x) \leq k\}\ , \quad C = \bigcup_k C_k\ .$$

(We have dropped the subscript $\mathcal{L}^1$ in $\underline{D}$.) We claim that $\mu_k(C) = 0$ for every k. Indeed, applying Lemma 6.5 to the set $C_k \cap A^*$, we obtain

$$\mu(C_k) = \mu(C_k \cap A^*) \leq k\mathcal{L}^1(C_k \cap A^*) \leq k\mathcal{L}^1(A^*) = 0\ .$$

It follows that $\mu(C) = 0$. Therefore, for μ-a.e. x, $\underline{D}\mu(x) = \infty$. Using definition,

$$\frac{\mu[x - \delta_n, x + \delta_n]}{2\delta_n} \to \infty,$$

for some $\delta_n \to 0$. In other words,

$$\frac{f(x+\delta_n) - f(x-\delta_n) + \mu(\{x+\delta_n\})}{2\delta_n} \to \infty \ .$$

Since μ is singular, even if $\mu(\{x+\delta_n\}) > 0$, we can find a point y arbitrarily close to $x+\delta_n$ such that $\mu(\{y\}) = 0$. In view of this, we may assume $\mu(\{x+\delta_n\}) = 0$, so

$$\frac{f(x+\delta_n) - f(x-\delta_n)}{2\delta_n} \to \infty \ , \qquad \text{as } n \to \infty \ .$$

On the other hand, if $f'_+(x)$ and $f'_-(x)$ are bounded, we have

$$f(x+\delta) = f(x) + f'_+(x)\delta + \circ(\delta) \ , \quad f(x-\delta) = f(x) - f'_-(x)\delta + \circ(\delta) \ ,$$

which implies

$$f(x+\delta_n) - f(x-\delta_n) = (f'_+(x) + f'_-(x))\delta_n + \circ(\delta_n),$$

so

$$\frac{f(x+\delta_n) - f(x-\delta_n)}{2\delta_n} \le |f'_+(x)| + |f'_-(x)| + 1 \ ,$$

for all large n, contradiction holds. We conclude either $f'_+(x)$ or $f_-(x)$ must blow up.

Note. See [R1] theorem 7.15 for a related result.

(10) Construct a continuous monotonic function F or $\mathbb{R}^1$ so that F is not constant on any segment although $F'(x) = 0$ a.e.

Solution. Let

$$A_n = \left\{ \frac{k}{2^n} : \ k = 0, 1, \cdots, 2^n \right\} \subset [0,1], \quad n \ge 0 \ , \quad A = \bigcup_n A_n \ .$$

Here A is the set of all rational binary numbers. In the following we define a sequence of continuous, piecewise functions F_n by assigning their values at A_n. First, define $F_0(x) = x$ so that $F_0(0) = 0$ and $F_0(1) = 1$. Assuming $F_{n-1}(x)$ has been defined for $x \in A_{n-1}$, $F_n(x)$ is defined as follows, if $x = 2k/2^n$, then $F_n(x) = F_{n-1}(k/2^{n-1})$ and, if $x = (2k+1)/2^n$, then

$$F_n\left(\frac{2k+1}{2^n}\right) = \frac{1}{4}F_{n-1}\left(\frac{k}{2^{n-1}}\right) + \frac{3}{4}F_{n-1}\left(\frac{k+1}{2^{n-1}}\right) .$$

At this point you better sketch the graphs of the first several F_n. Keep in mind that whenever $k/2^n$ appears in some previous A_m, say, $k/2^n = j/2^m, m < n$, $F_n(k/2^n) = F_m(j/2^m)$. You can see that each F_n is strictly increasing, $F_n(x) < F_{n+1}(x)$ for all $x \in (0,1)$, so that

$$F(x) = \lim_{n\to\infty} F_n(x) = \sup_n F_n(x)$$

is well-defined on $[0,1]$. Clearly $0 \le F(x) \le 1$ and $F(x) = F_n(x)$ for $x = k/2^n$.

Claim 1: F is strictly increasing. For, let $x < y$, we can find a large n and some k so that

$$x < \frac{k}{2^n} < \frac{k+1}{2^n} < y ,$$

so

$$F(x) \le F\left(\frac{k}{2^n}\right) = F_n\left(\frac{k}{2^n}\right) < F_n\left(\frac{k+1}{2^n}\right) = F\left(\frac{k+1}{2^n}\right) \le F(y) .$$

Claim 2: F is continuous. Consider $2k/2^n < (2k+1)/2^n < (2k+2)/2^n$. We have

$$\begin{aligned} F\left(\frac{2k+1}{2^n}\right) - F\left(\frac{2k}{2^n}\right) &= \frac{1}{4}F\left(\frac{k}{2^{n-1}}\right) + \frac{3}{4}F\left(\frac{k+1}{2^{n-1}}\right) - F\left(\frac{2k}{2^n}\right) \\ &= \frac{3}{4}\left(F\left(\frac{k+1}{2^{n-1}}\right) - F\left(\frac{k}{2^{n-1}}\right)\right) , \end{aligned}$$

and

$$\begin{aligned} F\left(\frac{2k+2}{2^n}\right) - F\left(\frac{2k+1}{2^n}\right) &= F\left(\frac{k+1}{2^{n-1}}\right) - \left(\frac{1}{4}F\left(\frac{k}{2^{n-1}}\right) + \frac{3}{4}F\left(\frac{k+1}{2^{n-1}}\right)\right) \\ &= \frac{1}{4}\left(F\left(\frac{k+1}{2^{n-1}}\right) - F\left(\frac{k}{2^{n-1}}\right)\right) . \end{aligned}$$

Therefore, for any two consecutive binary rational numbers in the same A_n,

$$\left| F\left(\frac{2k+1}{2^n}\right) - F\left(\frac{2k}{2^n}\right) \right| \leq \left(\frac{3}{4}\right)^n . \tag{1}$$

Now, if F is discontinuous, as an increasing function, it must be a jump discontinuity. At such x, $F(x^+) - F(x^-) \geq \rho_0 > 0$ for some ρ. However, for each n we can find some $k = 0, \cdots, 2^n$ such that $k/2^n \leq x < (k+1)/2^n$ or $k/2^n < x \leq (k+1)/2^n$. In view of (1),

$$F(x^+) - F(x^-) = \lim_{n\to\infty} \left(F\left(\frac{k+1}{2^n}\right) - F\left(\frac{k}{2^n}\right) \right) \to 0 ,$$

contradiction holds. Hence F must be continuous.

According to general theory, F is differentiable almost everywhere. Let I be the collection of all binary irrational numbers, that is, $x \in I$ if its binary expansion contains infinitely many 0 and 1. It is a set of full measure. Therefore, the set of all binary irrational numbers at which F is differentiable is also a set of full measure. Let us denote it by J.

Claim 3: $F'(x) = 0$ for $x \in J$. First, we observe that for $x \in J$, there exist binary rational numbers $\alpha_n,\ \beta_n$, where $\beta_n = \alpha_n + 1/2^n$ satisfying

$$\alpha_n < x < \beta_n ,$$

for all n. Moreover,

$$\alpha_n = \frac{z_1}{2} + \cdots + \frac{z_n}{2^n} , \quad z_j \in \{0, 1\} ,$$

and one must have either (a) $\alpha_n = \alpha_{n-1},\ \beta_n = \beta_{n-1} + 1/2^n$, or (b) $\alpha_n = \alpha_{n-1} + 1/2^n,\ \beta_n = \beta_{n-1}$. A review on the construction of the approximation to x by plotting the first several steps will convince you these facts.

In the case (a), we have

$$F(\beta_n) - F(\alpha_n) = \frac{1}{4}\left(F(\alpha_{n-1}) + \frac{3}{4}F(\beta_{n-1})\right) - F(\alpha_{n-1})$$
$$= \frac{3}{4}\left(F(\beta_{n-1}) - F(\alpha_{n-1})\right) \ ,$$

and, in the case (b),

$$F(\beta_n) - F(\alpha_n) = F(\beta_{n-1}) - \left(\frac{1}{4}F(\alpha_{n-1}) + \frac{3}{4}F(\beta_{n-1})\right)$$
$$= \frac{1}{4}\left(F(\beta_{n-1}) - F(\alpha_{n-1})\right) \ .$$

As F is differentiable and increasing, $F'(x) \in [0, \infty)$. If $F'(x) > 0$, the sequence

$$a_n = \frac{F(\alpha_n) - F(\beta_n)}{\beta_n - \alpha_n} \to F'(x) \ , \qquad \text{as } n \to \infty \ .$$

It follows that $a_n/a_{n-1} \to 1$ as $n \to \infty$. However, the relations above tell us that $a_n/a_{n-1} = 1/2$ or $3/2$, which never converges to 1. Hence $F'(x)$ must vanish.

Note. This example is a special case of an example in 18.6 in [HS].